Maximize $Z = 15x_1+30x_2+4x_1x_2-2x_1^2-4x_2^2$

Subject to $x_1+2x_2 \leq 30$,

$x_1, x_2 \geq 0$

solution:

standard notation and representation :

maximize $Cx - \frac{1}{2}x^T Qx$

subject to $Ax \leq b$

$x \geq 0$

converting whole expression in standard notation and representation:

=>

To put the constraints in our standard form of $Ax \leq b$ we have

A = [1, 2]

x = | x1 |

| x2 |

b = [30]

- ⇨ Ax<=b
- ⇨ So, [1, 2] | x1 | ≤ [30]

| x2 |

To put the objective

$15x1 + 30x2 + 4x1x2 - 2(x1)^2 - 4(x2)^2$

in our standard form of Cx - ½x T Qx we have:

c = [15, 30]

Q = | 4 -4 |

| -4 8 |

Here to get Q we double the coefficients on the nonlinear terms and change the signs (note the - ½ in - ½x T Qx).

$$x^T Qx = [x_1, x_2] \begin{vmatrix} 4 & -4 \\ -4 & 8 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$$= [x_1, x_2] \begin{vmatrix} 4x_1 - 4x_2 \\ -4x_1 + 8x_2 \end{vmatrix}$$

$$= 4(x_1)^2 - 4x_1x_2 - 4x_1x_2 + 8(x_2)^2$$

$$= 4(x_1)^2 - 8x_1x_2 + 8(x_2)^2$$

so $cx - \frac{1}{2}x^T Qx =$

$$[15, 30] \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} - \frac{1}{2}(4(x_1)^2 - 8x_1x_2 + 8(x_2)^2)$$

$$= 15x_1 + 30x_2 - \frac{1}{2}(4(x_1)^2 - 8x_1x_2 + 8(x_2)^2)$$

$$= 15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

as required So we have the problem expressed in standard form, where note that Q is symmetric.

For our QP we have
$f = 15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$
with just one constraint (so m=1), namely
$g_1 = x_1 + 2x_2 \leq b_1 = 30$

$\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j) \leq 0$ at $x_j=X_j\ \forall j$
becomes
for j=1 $\quad 15 + 4X_2 - 4X_1 - u_1 \leq 0$
for j=2 $\quad 30 + 4X_1 - 8X_2 - 2u_1 \leq 0$

$X_j[\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j)] = 0$ at $x_j=X_j\ \forall j$
becomes
for j=1 $\quad X_1[15 + 4X_2 - 4X_1 - u_1] = 0$
for j=2 $\quad X_2[30 + 4X_1 - 8X_2 - 2u_1] = 0$

$g_i(X) - b_i \leq 0\ \forall i$ becomes
$X_1 + 2X_2 - 30 \leq 0$

$u_i[g_i(X) - b_i] = 0\ \forall i$ becomes
$u_1(X_1 + 2X_2 - 30) = 0$

where $X_1, X_2, u_1 \geq 0$

$4X_2 - 4X_1 - u_1 + y_1 = -15$
$4X_1 - 8X_2 - 2u_1 + y_2 = -30$
$X_1 + 2X_2 + v_1 = 30$

Now consider the three constraints that require a product to be zero, these are

$X_1[15 + 4X_2 - 4X_1 - u_1] = 0$
$X_2[30 + 4X_1 - 8X_2 - 2u_1] = 0$
$u_1(X_1 + 2X_2 - 30) = 0$

which using the above becomes

$X_1[-y_1] = 0$
$X_2[-y_2] = 0$
$u_1(-v_1) = 0$

The initial simplex tableau is:

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1	4	-4	1	-1			1		15
z_2	-4	8	2		-1			1	30
v_1	1	2				1			30
Obj		-4	-3	1	1				-45

Summarising, the pivot row is the z_2 row; the pivot element is 8; the pivot column is the X_2 column.

The new simplex tableau is

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1	2		2	-1	-0.5		1	0.5	30
X_2	-0.5	1	0.25		-0.125			0.125	3.75
v_1	2		-0.5		0.25	1		-0.25	22.5
Obj	-2		-2	1	0.5			0.5	-30

Doing the pivot operation we get (to 3 decimal places)

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1			2.5	-1	-0.75	-1	1	0.75	7.5
X_2		1	0.125		-0.063	0.25		0.063	9.375
X_1	1		-0.25		0.125	0.5		-0.125	11.25
Obj			-2.5	1	0.75	1		0.25	-7.5

Doing the pivot operation we get (to 3 decimal places)

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
u_1			1	-0.4	-0.3	-0.4	0.4	0.3	3
X_2		1		0.05	-0.025	0.3	-0.05	0.025	9
X_1	1			-0.1	0.05	0.4	0.1	-0.05	12
Obj							1	1	0

Here we are done, as the objective has value zero. Hence we have a solution $u_1=3$, $X_2=9$ and $X_1=12$, all other variables zero.

It is easy to confirm that this solution satisfies:

$$-4X_2 + 4X_1 + u_1 - y_1 + z_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 + z_2 = 30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
$$\text{all variables} \geq 0$$

As we have a solution satisfying our KKT constraints this must be the optimal solution to our original QP

maximise

$$15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

subject to

$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

i.e. the optimal solution to this problem is $x_1=12$ and $x_2=9$, for which the associated objective function value is 270.